wind turbines and the design of high buildings) often require the extrapolation of the wind profile over considerable height intervals. Fore these purposes the two types of description are only equivalent if it is possible to find a power law that fits to the logarithmic profile in slope and curvature over the respective range. The following investigation will demonstrate that this is possible only for certain combinations of surface roughness and atmospheric stability in stably stratified boundary-layer flow.

Basic Profile Equations

The power law is usually formulated:

$$u(z) = u(z_A)\,(z/z_A)^n \qquad (1)$$

with the anemometer height z_A and the exponent n. The logarithmic law reads for neutral stability:

$$u(z) = (u_*/\kappa)\,\ln(z/z_0) \qquad (2)$$

with the friction velocity u_*, the von Kármán constant $\kappa = 0.4$. For non-neutral stratification (2) is modified to:

$$u(z) = (u_*/\kappa)\,(\ln(z/z_0) - \psi(z/L_*)) \qquad (3)$$

with

$$\psi(z/L_*) = \begin{cases} \ln((1+x^2)/2((1+x)/2)^2) - 2\,\mathrm{arctg}(x) + \pi/2 & \text{for } z/L_* < 0 \\ -4.7\,z/L_* & \text{for } z/L_* \geq 0 \end{cases} \qquad (4)$$

and $x = (1 - 15\,z/L_*)^{1/4}$. The values for the two constants 4.7 and 15 (we use here the values given by Businger [5]) vary slightly in the literature. The following comparison of the power law with the logarithmic profiles does not depend on the exact values of these constants. The Monin-Obukhov-length L_* is defined by:

$$L_* = -\rho c_p \Theta u_*^3 / (\kappa g H_0) \qquad (5)$$

with air density ρ, the specific heat of air c_p, the potential temperature Θ, the Earth's gravity acceleration g, and the surface turbulent heat flux H_0.

In the next section we will consider the most simple case of neutrally stratified flow by comparing (1) and (2). After this, in the subsequent section, we will analyse the general case of non-neutrally stratified flow by comparing (1) and (3).

Comparison of the Two Profile Laws for Neutral Stratification

Two wind profiles are identical if they have equal slope and curvature in all heights. They are nearly identical in a small height interval if they have equal slope in the center of the height interval. The height interval where we find an approximate sameness of the two profiles would be larger if not only the slope but also the curvature is identical in the center of this interval. While we can always find parameter sets that make the slopes of (1) and (2) identical at a given height it is not guaranteed that the curvature can be made equal, too. For the investigation of the possibility whether this can happen we need the mathematical formulation of the slope and the curvature of (1) and (2).

The slope of the logarithmic wind profile under neutral stratification (2) is given by:

$$\partial u/\partial z = (1/\kappa)\,(u_*/z) = \ln^{-1}(z/z_0)\,u(z) / z \qquad (6)$$

and the curvature of the logarithmic profile follows by taking the second derivative of (2):

$$\partial^2 u/\partial^2 z = -(1/\kappa)\,(u_*/z^2) = -\ln^{-1}(z/z_0)\,u(z) / z^2 \qquad (7)$$

The slope of the power law (1) by differentiating yields:

$$\partial u/\partial z = u(z_A) / z_A\, n\,(z/z_A)^n / (z / z_A) = n\,u(z_A)\,(z/z_A)^n / z = n\,u(z) / z \qquad (8)$$

and the curvature of the power laws reads:

$$\partial^2 u/\partial^2 z = n\,(n-1)\,u(z_A)\,(z/z_A)^n / z^2 = n\,(n-1)\,u(z) / z^2 \qquad (9)$$

Equating the slopes (6) and (8) delivers:

$$n = \ln^{-1}(z/z_0) \qquad (10)$$

which equals the formulation given by Sedefian [4] in the limit of neutral stratification. (10) means that the exponent n decreases with height for a given roughness length z_0. The height in which the slopes of the two wind profiles (1) and (2) should be equal – this is usually the anemometer height $z = z_A$ – has therefore to be specified a priori. The dependence of n on height is the stronger the smaller the ratio z/z_0 is (see Fig. 1). Due to this fact the dependence of n with height is stronger for complex terrain where the roughness length z_0 is large and it can nearly be neglected for water surfaces with very small roughness lengths. In order to see whether we can find an exponent n so that both the slope and the curvature agree in a given height we must equate the formulas for the curvature of the two profiles (7) and (9). This yields the relation:

$$n\,(n-1) = -\ln^{-1}(z/z_0) \qquad (11)$$

height in order to find a power law profile with the same slope and curvature as the logarithmic profile. The curved thin lines from the lower left to the upper right represent the solution of equation (14), the lines with the maximum just left of $z/L_* = 0$ the solution of equation (15) (please note that the lowest line is the one for n = 0.5, and that the lines for n = 0.3 and n = 0.7 are identical), and the thick line marks the solution of (17). As designed the thick curve goes through the points where solutions from (14) and (15) are identical.

Fig. 4 displays three examples of wind profiles for non-neutral stratification, one for unstable conditions and a large roughness length, one which lies exactly on the curve from equation (17) so that slope and curvature coincide simultaneously, and one for very stable conditions. For a roughness length of z_0 = 0.023 m (z/z_0 = 2173) and a Monin-Obukhov-length of L_* = 1500 m (z/L_* = 0.0333) a power law profile with n = 0,15 has equal slope and curvature at $z = z_A$ = 50 m as the logarithmic profile. At z = 100 m the two profiles only differ by 0.1%, at 10 m by 0.9%. This is an even better fit than the fit for the neutral wind profile with z/z_0 = 5000 in Fig. 2. For the two profiles under unstable conditions the respective deviations at 100 m and at 10 m are 4.5% and 89.9%, for the two profiles under very stable conditions these deviations are -3.5% and -14.0%.

Conclusions

We have extended the analysis by Sedefian [4] and shown that only for certain conditions in stably stratified boundary-layer flow it is possible to find a power law profile that has the same slope and curvature as a logarithmic wind profile and thus fits the logarithmic profile almost perfectly over a wide height range. The respective combinations of roughness and Monin-Obukhov-length for which this good fit is possible have been derived analytically. In a purely neutrally stratified boundary-layer this perfect fit is not possible although the fit becomes the better the smoother the surface is. The worst fit occurs for unstable conditions and high roughness lengths.

For high wind speeds which are most favourable for wind energy conversion the stratification of the boundary-layer usually becomes nearly neutral. The above considerations then show that only for very smooth terrain (offshore and near the coasts) the power law is a good approximation to the real surface-layer wind profile. Extrapolations of the wind profile above the height of the surface layer (80 to 100 m) by either laws (1) or (3) should be made with very great care because these laws are valid for the surface layer only (Emeis [6]).

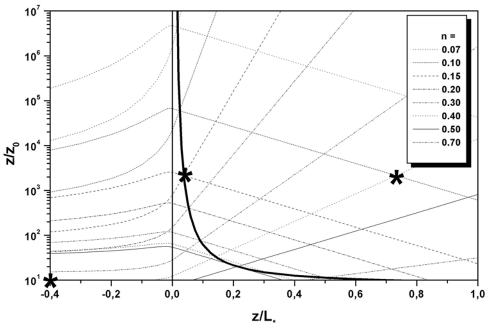

Fig. 3: Solution of the equations (14), (15), and (17) in the phase space spanned by the roughness parameter z/z_0 and the stratification parameter z/L_*. Thin lines from lower left to upper right (calculated from (14)) indicate for different exponents n when a logarithmic profile and a power law profile have equal slopes, thin lines from left to lower right (calculated from (15)) indicate for different exponents n when a logarithmic profile and a power law profile have equal curvatures, the thick line (calculated from (17)) runs through the points where the solutions from (14) and (15) are equal. The three asterisks mark the position of the examples shown in Fig. 4.

Due to the fact that the atmosphere is usually stably stratified in the mean it becomes obvious from the above calculations why the power law approach has been so successful in many cases.

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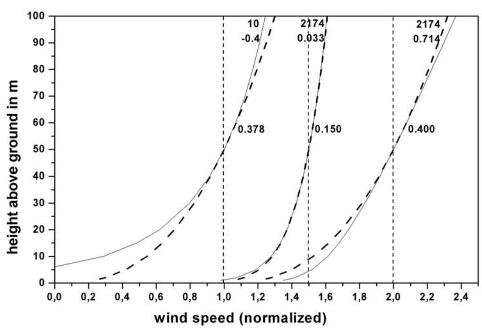

Fig. 4: Three logarithmic wind profiles for non-neutral stratification ($z/L_* \neq 0$) and their approximation by power law profiles. The middle pair of profiles has been shifted by 0.5, the rightmost pair by 1.0 to the right. The two numbers at the top of the profiles give z/z_0 and z/L_*, the number in the middle of the profiles the exponent n of the respective power law.